Mathematics 222B Lecture 19 Notes

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April 5, 2022

1 Decay by Dispersion for the Wave Equation

1.1 Motivation: the picture of decay by dispersion for the wave equation

Consider the wave equation in $\mathbb{R}^{1+d}$,

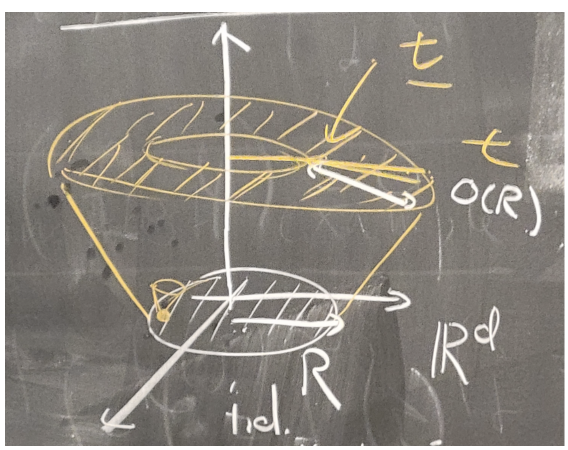
$$\Box\phi = 0, \qquad \Box = -\partial_t^2 + \Delta.$$

We know the conservation of energy:

$$\int ((\partial_t \phi)^2 + |D\phi|^2)|_{t=t_1}\, dx = \int ((\partial_t \phi)^2 + |D\phi|^2)|_{t=0}\, dx \qquad \forall t_1 \in \mathbb{R}.$$

In some sense, the size of ϕ stays constant. Since we are in an infinite dimensional space of functions, we may have a notion of size where the function stays the same in time and another notion of size where the function goes to 0 in time.

Dispersion is a *decay* mechanism for $\Box\phi = 0$ in $\mathbb{R}^{1+d}$, where the amplitude "$|\phi|(t) \to 0$ as $t \to \pm\infty$. " This pointwise estimate will not always hold, but this is the idea. Assume that the initial data is well-localized (compactly supported or at least has strong decay). The solution should try to propagate in every direction (forming a cone in $\mathbb{R}^{1+d}$). At time t, most of the solution should have spread away around $O(R)$.



The quantity $\int |\partial\phi|^2 \, dx$ is conserved. We normalize this so that $\int |\partial\phi|^2|_{t=0} \, dx = 1$ and $R = 1$. At time T, $\int |\partial\phi|^2|_{t=T} \, dx = 1$, and ϕ is supported (evenly) in $\{x : t < |x| < t+1\}$. This means that

$$1 = \int |\partial\phi|^2|_{t=T} \, dx \approx a^2 t^{d-1},$$

where a is the amplitude of ϕ at time t. This means that

$$a \approx \frac{1}{t^{(d-1)/2}}.$$

This is also the decay rate for $\|\partial\phi\|_{L^\infty}(t)$ and also for $\|\phi\|_{L^\infty}(t)$. The intuition for the latter statement is that if we know that ϕ is small near 0 at time t, then we can just integrate radially in constant time; this does not give so much up because $R = 1$.

Our goal is to make this decay precise. We will aim to give two proofs of this fact:

1. Using oscillatory integrals: This generalizes to constant foefficient dispersive PDEs such as $(\Box - m^2)\phi = f$ or $(\partial_t + \Delta)\phi = f$.

2. Vector field method: This generalizes better to variable coefficent PDEs and nonlinear PDEs.

1.2 Oscillatory integrals in the solution to the wave equation

The starting point is the solution formula for $\Box\phi = f$ using the Fourier transform. Take the spatial Fourier transform of the equation, $(-\partial_t^2 + \Delta)\phi = f$, which means $\widehat{\phi}(t,\xi) = \mathcal{F}_x[\phi(t,\cdot)]$. This gives

$$-\partial_t^2 \widehat{\phi}(t,\xi) - |\xi|^2(t,\xi) = \widehat{f}.$$

In view of Duhamel's formula, it suffices to consider $f = 0$. So we now have $\partial_t^2 \widehat{\phi} = -|\xi|^2 \widehat{\phi}$. This has the solution $e^{\pm it|\xi|}$, so

$$\widehat{\phi}(t,\xi) = a_+(\xi) e^{it|\xi|} + a_-(\xi) e^{-it|\xi|},$$

where a_+, a_- are determined from the initial conditions at $t = 0$. We can then write

$$\phi(t,x) = \frac{1}{(2\pi)^d} \int a_+(\xi) e^{it|\xi||} e^{ix\cdot\xi} \, d\xi + \frac{1}{(2\pi)^d} \int a_-(\xi) e^{-it|\xi||} e^{ix\cdot\xi} \, d\xi.$$

These integrals are essentially the same, so we will concentrate on the + case. Our goal is to analyze the asymptotics of this integral in (t,x).

1.3 General theory for oscillatory integrals

1.3.1 Principle of nonstationary phase

We now take an intermission to study some model oscillatory integrals.

Definition 1.1. An **oscillatory integral** is an integral of the form

$$I(\lambda) = \int_{-\infty}^{\infty} a(\xi) e^{i\lambda\Phi(\xi)}\, d\xi, \qquad \xi \in \mathbb{R}.$$

Here $a : \mathbb{R} \to \mathbb{C}$ is a the **amplitude function**, and $\Phi : \mathbb{R} \to \mathbb{R}$ is called the **phase function**. We assume a and ξ to be small, i.e. $|D^\alpha|$, $|D^\alpha \Phi| \lesssim_\alpha 1$. We also assume that $\operatorname{supp} a$ is compact.

Proposition 1.1 (Principle of nonstationary phase)**.** *If* $|\partial_\xi \Phi| \geq \eta$ *on* $\operatorname{supp} a$, *then*

$$I(\lambda)| \lesssim_k \frac{1}{\lambda^k}.$$

The idea is that the oscillations will make a lot of cancelation, so the size of the integral will be much smaller than if we just integrated a.

Proof. Use integration by parts; the key identity that drives this is $\partial_\xi(e^{i\lambda\Phi(\xi)}) = i\lambda\partial_\xi\Phi(\xi)e^{i\lambda\Phi}$, which gives $e^{i\lambda\Phi} = \frac{1}{i\lambda\partial_\xi\Phi}\partial_\xi(e^{i\lambda\Phi})$. This gives the identity

$$\begin{aligned} I(\lambda) &= \int a(\xi)\frac{1}{i\lambda\partial_\xi\Phi(\xi)}\partial_\xi e^{i\lambda\Phi}\, d\xi \\ &= -\int \partial_\xi \left(a(\xi)\frac{1}{i\lambda\partial_\xi\Phi(\xi)}\right) e^{i\lambda\Phi}\, d\xi. \end{aligned}$$

This is good, as long as $|\partial_\xi\Phi| \geq \eta_0$. The derivative part is $\partial_\xi a \frac{1}{i\lambda\partial_\xi\Phi} - a\frac{1}{i\lambda\partial_\xi\Phi}\frac{\partial_\xi^2\Phi}{\partial_\xi\Phi}$.

$$\lesssim \frac{1}{\lambda}.$$

Integrating by parts k times gives $|I(\lambda)| \leq \cdots \lesssim \frac{1}{\lambda^k}$. □

1.3.2 Principle of stationary phase

In the presence of a critical point ξ_0 of Φ (i.e. $\partial_\xi\Phi(\xi_0) = 0$), we have the **principle of stationary phase** Consider

$$I(\lambda) = \int a(\xi)e^{i\lambda\Phi}\, d\xi$$

with $\partial_\xi \Phi(0) = 0$ and no other zeros in the support of a. (The general case can be reduced to this by a smooth partition of unity.) In view of Taylor expansion, we would expect that

$$\Phi(\xi) = \Phi_0 + c\xi^n + \cdots, \qquad \text{where } n \geq 2.$$

We can absorb $e^{i\lambda\Phi_0}$ into a, so we may assume that $\Phi_0 = 0$. So our model case is when $\Phi(\xi) = c_n\xi^n$. Here,

$$I(\lambda) = \int a(\xi) e^{ic_n\lambda\xi^n} \, d\xi.$$

The principle is that the **stationary phase region** $\{\xi \in \mathbb{R}^n : |\lambda\xi^n| \leq 1\}$ gives you the main contribution:

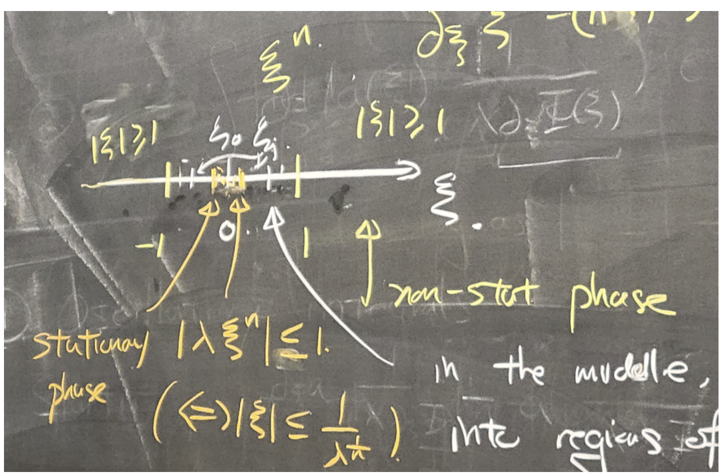
$$I(\lambda) \sim \int_{\{|\lambda\xi^n| < 1\}} d\xi \approx a(0)C\frac{1}{\lambda^{1/n}},$$

where $\frac{1}{\lambda^{1/n}}$ is the volume of the region $\{|\xi| \leq 1/\lambda^{1/n}\}$.

How do we make this precise? Here are two approaches:

1. Precise algebraic manipulation.
   (a) Change of variables
   (b) Use the Fourier transform of the Gaussian ($n = 2$).
2. (Less precise but more robust) Dyadic decomposition.

We will present the latter approach. In the two regions with $|\xi| \geq 1$, we can use the principle of non-stationary phase. On the other hand, we have stationary phase in the region very close to 0, where $|\xi| \leq 1/\lambda^{1/n}$.



The idea is that in the middle, we can decompose into regions of the form $A_j\{2^{j-1} \leq |\lambda\xi^n| \leq 2^j\}$ for $j \geq 1$. In each of these regions $\lambda\xi^n$ is roughly constant. In particular, we introduce a smooth partition of unity $\{\zeta_j\}_j$ subordinate to $\{A_j\}_j$, and use two estimates for the integral

$$I_j \int \zeta_j(\xi)a(\xi)e^{i\lambda\Phi}\,d\xi,$$

Here, we can use the estimate of stationary phase (ignore $e^{i\cdots}$) and the estimate of nonstationary phase.

Here are the details. Let ζ_0 be adapted to $\{|\lambda\xi^n| \leq 1$. We have

$$I(\lambda) = \underbrace{\int \xi_0 a e^{i\lambda\xi^n}\,d\xi}_{I_0} + I_j.$$

Then

$$|I_0| \lesssim \frac{1}{\lambda^{1/n}}.$$

For I_j, if we do not integrate by parts, we get

$$|I_j| \lesssim |A_j| \lesssim (2^{1/n})^j \lambda^{-1/n},$$

where we have used $A_j \subseteq \{|\xi| \leq 2^{j/n}/\lambda^{1/n}\}$. This is nice when j is small but bad when j is big. If we use integration by parts, we get

$$|I_j| = \int \left|\partial_\xi\left(\zeta_j(\xi)a(\xi)\frac{1}{i\lambda\partial_\xi\Phi}\right)e^{i\lambda\Phi}\right|\,d\xi.$$

Note that $|\lambda\partial_\xi\Phi| \simeq e^j\frac{1}{|\xi|} \simeq 2^{(1-1/n)j}\lambda^{1/n}$. We also have $|a| + |\partial_\xi a| \lesssim 1$,

$$\left|\frac{\partial_\xi^2\Phi}{\partial_\xi\Phi}\right| \lesssim \frac{1}{|\xi|} \approx 2^{-j/n}\lambda^{1/n}, \qquad |\partial_\xi\zeta_j| \lesssim \frac{1}{|\xi|} \lesssim 2^{-1/n}\lambda^{1/n}.$$

So

$$\left|\zeta_j(\xi)a(\xi)\frac{1}{i\lambda\partial_\xi\Phi(\xi)}\right| \sim 2^{(-1+1/n)j}\lambda^{-1/n},$$

$$\begin{aligned}
|\partial_\xi(\cdots)| &\lesssim \frac{1}{|\xi|}2^{(-1+1/n)j}\lambda^{-1/n} \\
&\lesssim 2^{-j/n}\lambda^{1/n}2^{(-1+1/n)j}\lambda^{-1/n} \\
&= 2^{-j}.
\end{aligned}$$

So

$$\int || \, d\xi| \lesssim 2^{-j} 2^{j/n} \lambda^{-1/n} = 2^{-(1-1/n)j} \lambda^{-1/n}.$$

Putting the these bounds for each $|I_j|$ together, we get

$$\left| \sum_{g \geq 1} I_j \right| \lesssim \sum_{j \geq 1} 2^{-(1-1/n)j} \lambda^{-1/n} \lesssim \lambda^{-1/n}.$$

Remark 1.1. We did not use both our bounds at the end. This is because we picked our dyadic decomposition in a smart way. If we had picked $\widetilde{A}_j = \{|\xi| \simeq 2^j\}$, then we would still be able to proceed with the proof, but we would need both the integration by parts and non-IBP bound.